

Sampling with Mirrored Stein Operators

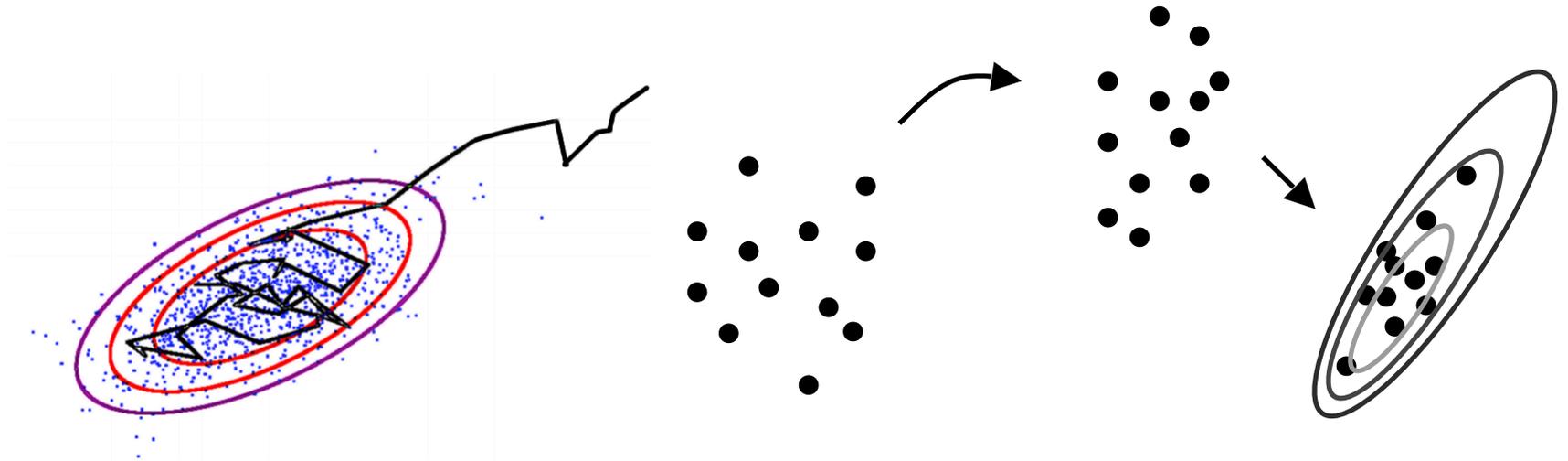
Jiaxin Shi

Microsoft Research New England

Joint work with Chang Liu, Lester Mackey

Sampling from an Unnormalized Distribution

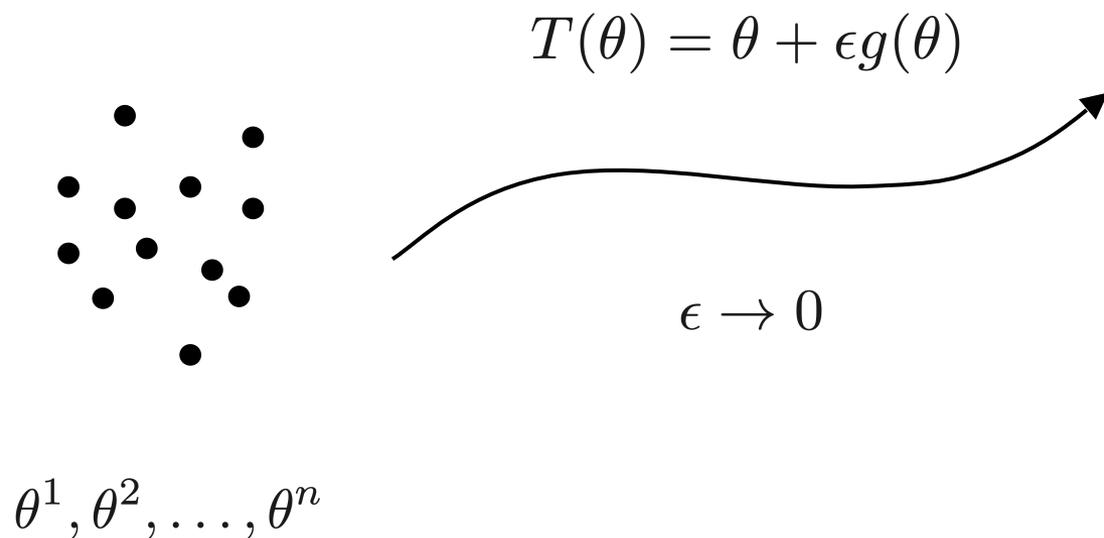
Fig. from Murray (2009)



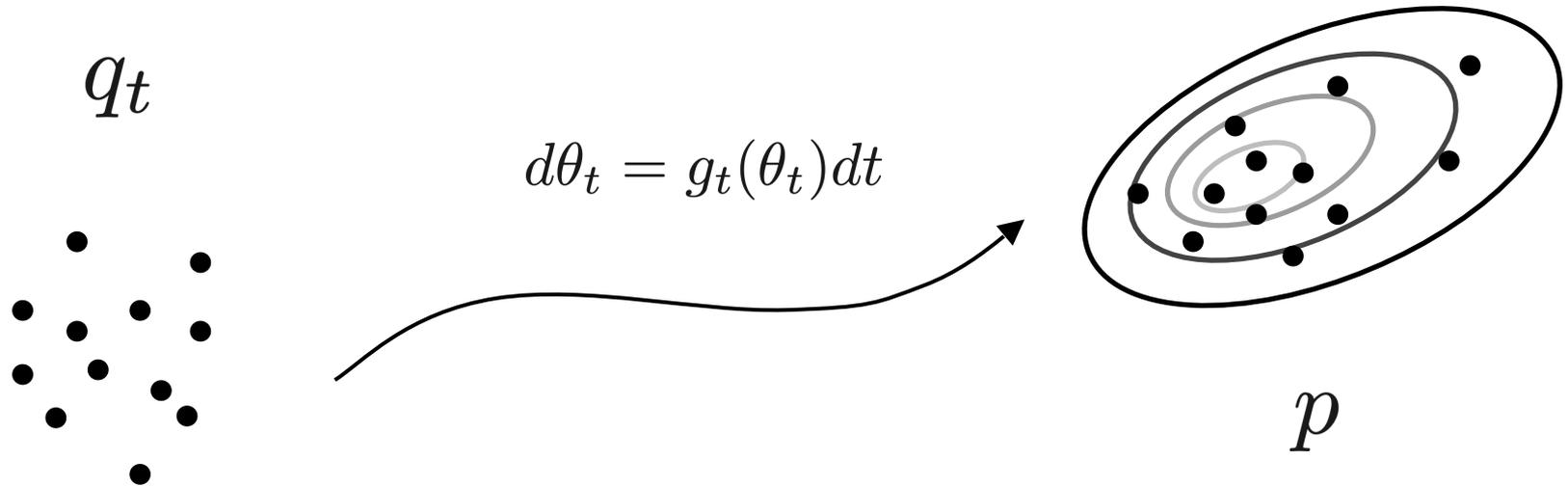
MCMC

Particle evolution methods
e.g., Stein Variational Gradient Descent

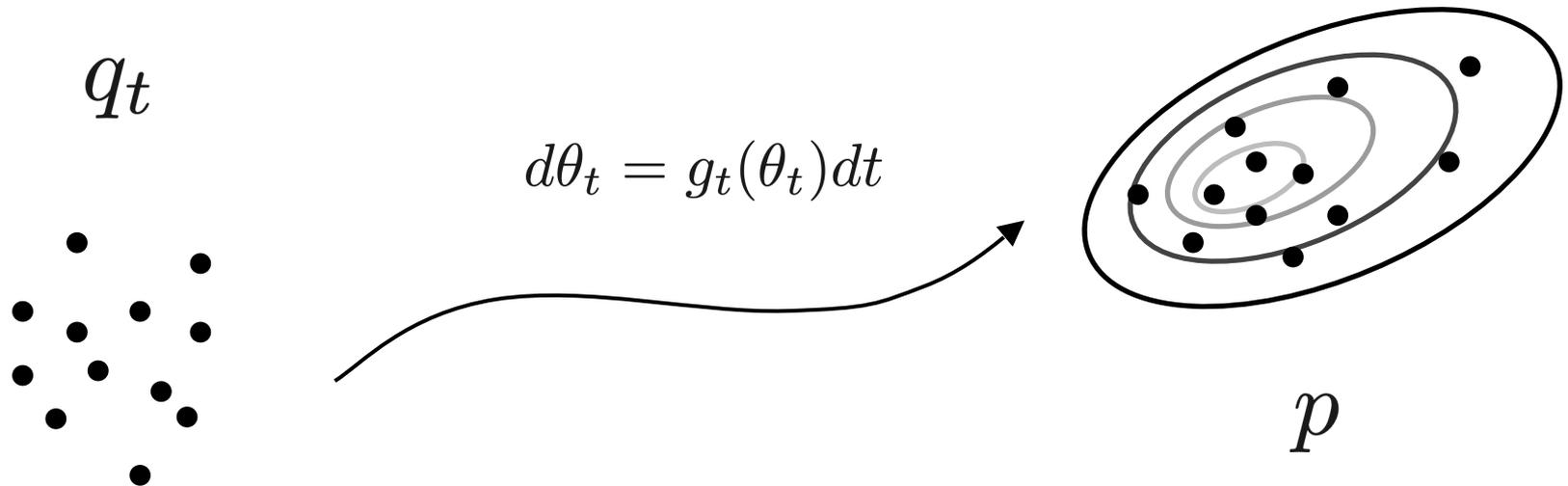
Stein Variational Gradient Descent (SVGD)



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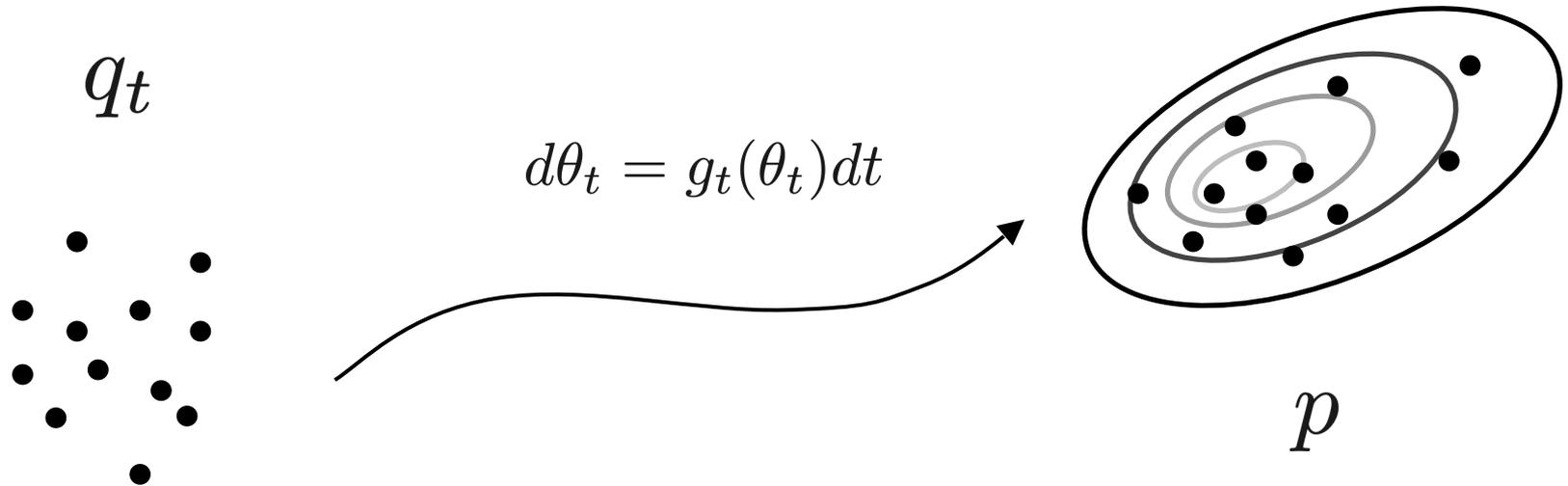
Stein Variational Gradient Descent (SVGD)



(Liu & Wang, 2016)

$$\frac{d}{dt} \text{KL}(q_t \| p) = -\mathbb{E}_{q_t} [(\mathcal{S}_p g_t)(\theta)]$$

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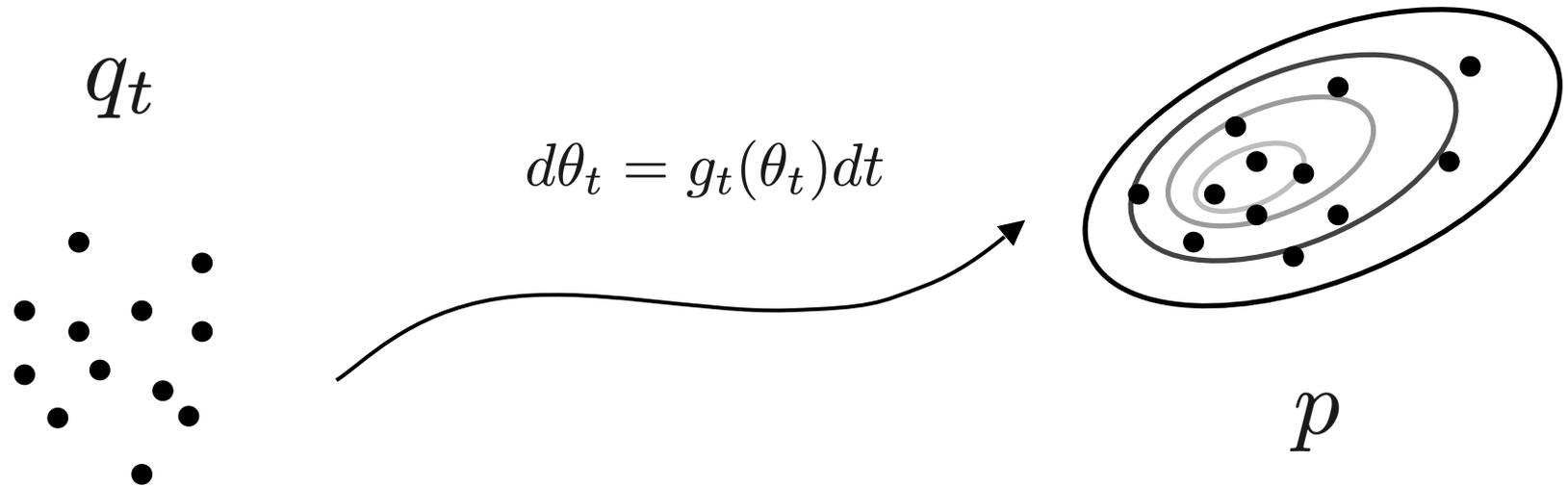
(Gorham & Mackey, 2015)

Langevin Stein Operator: $(\mathcal{S}_p g)(\theta) = g(\theta)^\top \nabla \log p(\theta) + \nabla \cdot g(\theta)$

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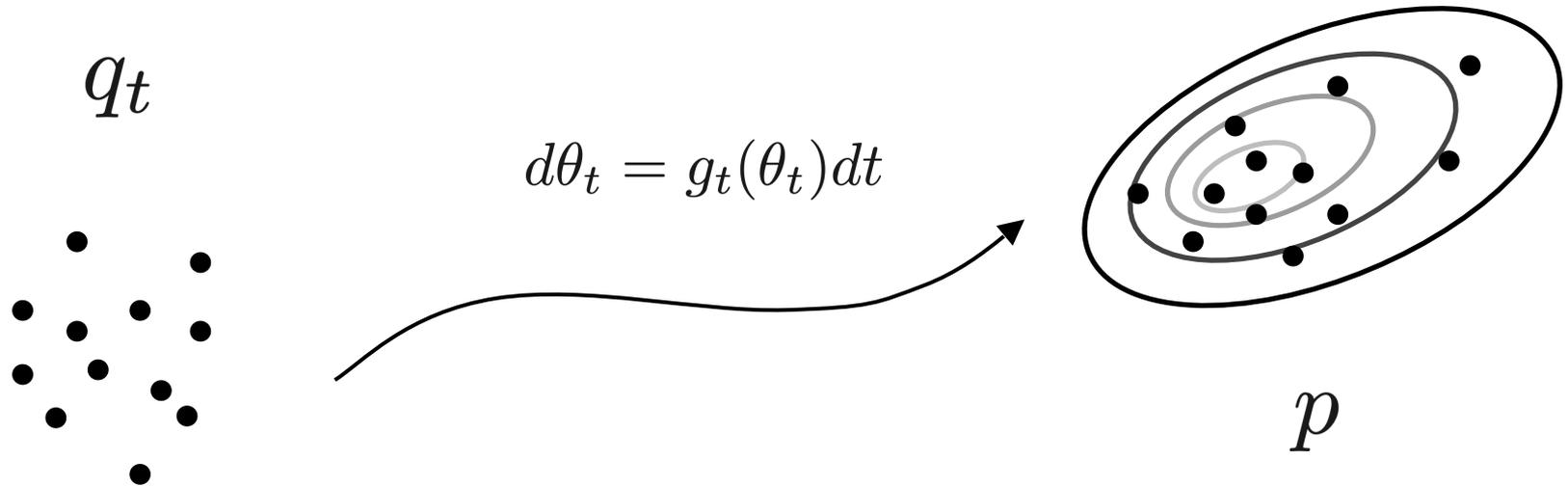
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$$\frac{d}{dt} \text{KL}(q_t \| p) = -\mathbb{E}_{q_t} [(\mathcal{S}_p g_t)(\theta)]$$

Find the direction that **most quickly** decreases the KL divergence to p

Stein Variational Gradient Descent (SVGD)



(Gorham & Mackey, 2015)

Langevin Stein Operator: $(\mathcal{S}_p g)(\theta) = g(\theta)^\top \nabla \log p(\theta) + \nabla \cdot g(\theta)$

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$$g_t^* = \arg \min_{g_t \in \mathcal{H}, \|g_t\|_{\mathcal{H}} \leq 1} \frac{d}{dt} \text{KL}(q_t \| p) \propto \mathbb{E}_{q_t} [\mathcal{S}_p K(\cdot, \theta)]$$

Optimal direction in the RKHS of kernel K .

Two Regimes of SVGD

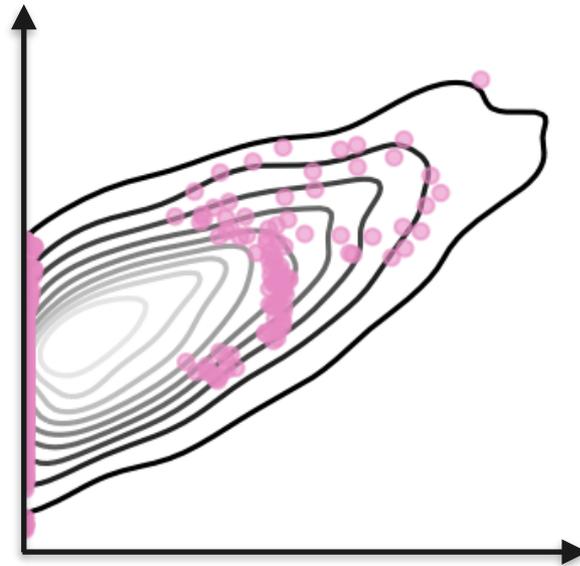
(Liu & Wang, 2016)

$$\theta_{t+1}^i \leftarrow \theta_t^i + \epsilon_t \frac{1}{n} \sum_{j=1}^n \left(K(\theta_t^i, \theta_t^j) \nabla \log p(\theta_t^j) + \nabla_{\theta_t^j} \cdot K(\theta_t^j, \theta_t^i) \right)$$

- $n = 1$: reduces to gradient descent on $-\log p(\theta)$ if $\nabla \cdot K(\theta, \theta) = 0$.
- $n \rightarrow \infty$: weak convergence to p under certain conditions.

(Gorham & Mackey, 2017; Liu 2017; Gorham et al., 2020)

They Break Down for Constrained Targets



SVGD + Projection: Samples end up collecting on the boundary.

Langevin Stein Operators

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The last identity holds because of divergence theorem:

$$\int_{\Theta} \nabla \cdot ((p(\theta)g(\theta))d\theta = 0 \Leftrightarrow \int_{\partial\Theta} p(\theta)g(\theta)^\top n(\theta)d\theta = 0$$

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Therefore, $q_t = p$ is a stationary point of the SVGD dynamics.

Two Problems of SVGD for Constrained Targets

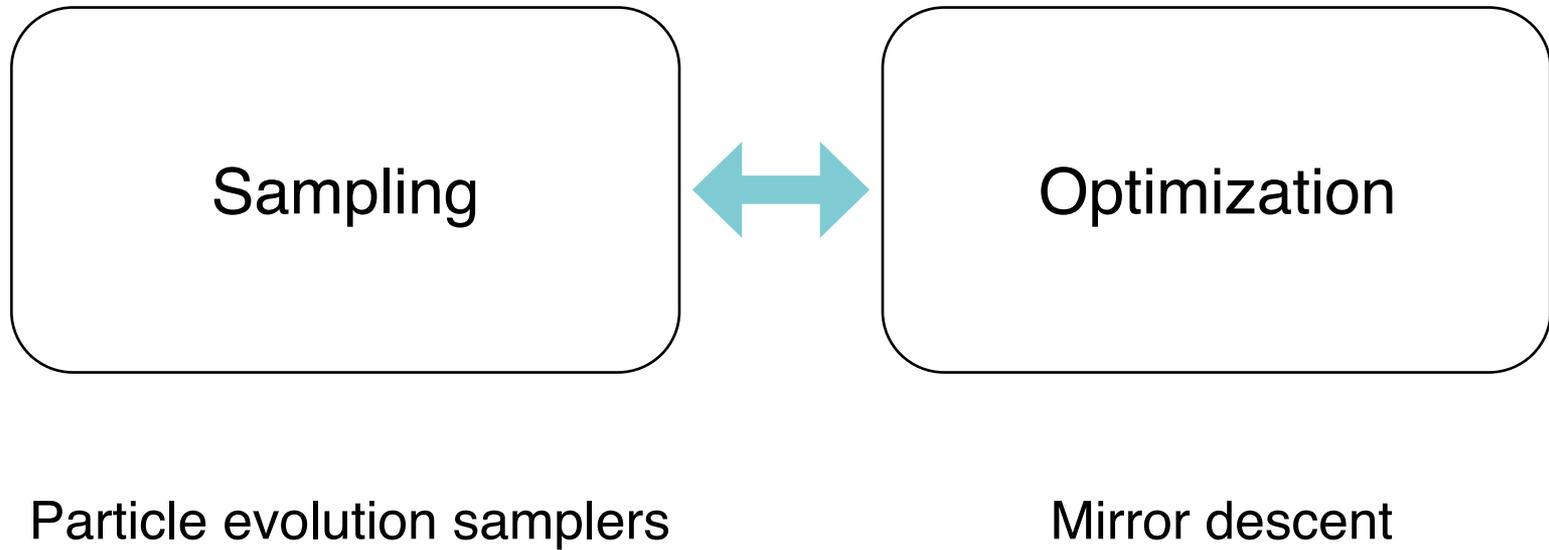
Two Problems of SVGD for Constrained Targets

- Standard SVGD updates can push the particles outside of its support
 - Result: Future updates undefined.

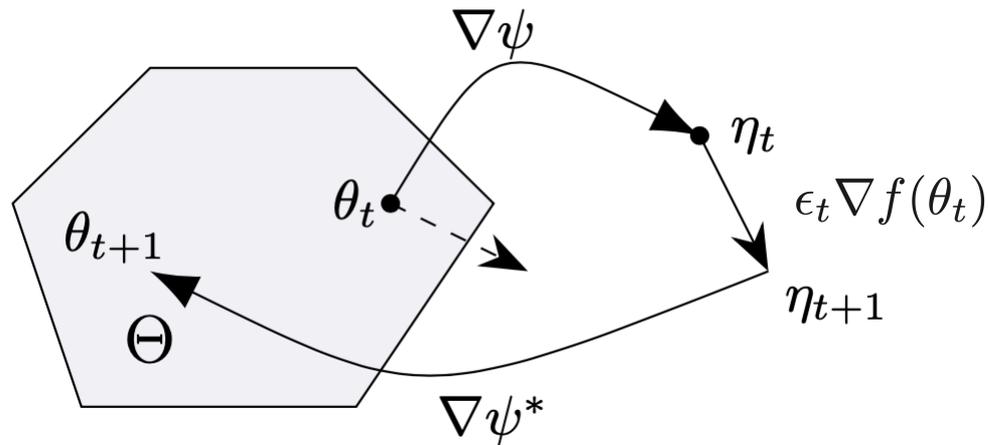
Two Problems of SVGD for Constrained Targets

- Standard SVGD updates can push the particles outside of its support
 - Result: Future updates undefined.
- The boundary conditions may fail to hold for g in the RKHS
 - This happens when p is non-vanishing or explosive on the boundary
 - Result: SVGD need not converge to p since p is not a stationary point.

This Talk is About



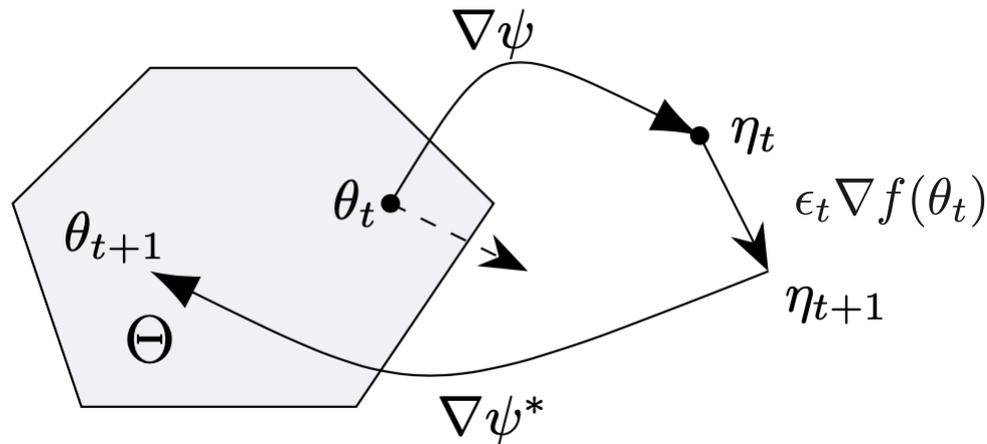
Mirror Descent



Strictly convex $\psi : \Theta \rightarrow \mathbb{R} \cup \{\infty\}$

$$(\nabla\psi)^{-1} = \nabla\psi^*$$

Mirror Descent



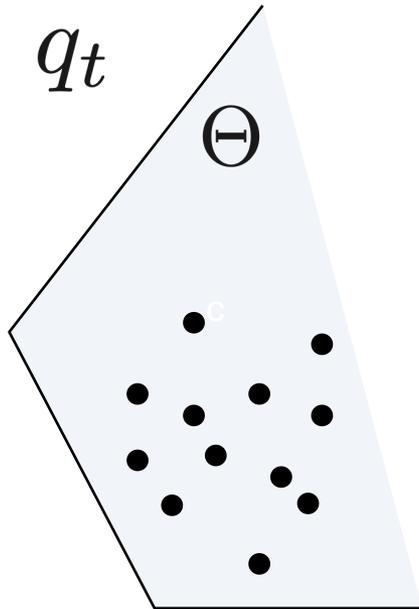
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Continuous time limit: mirror flow

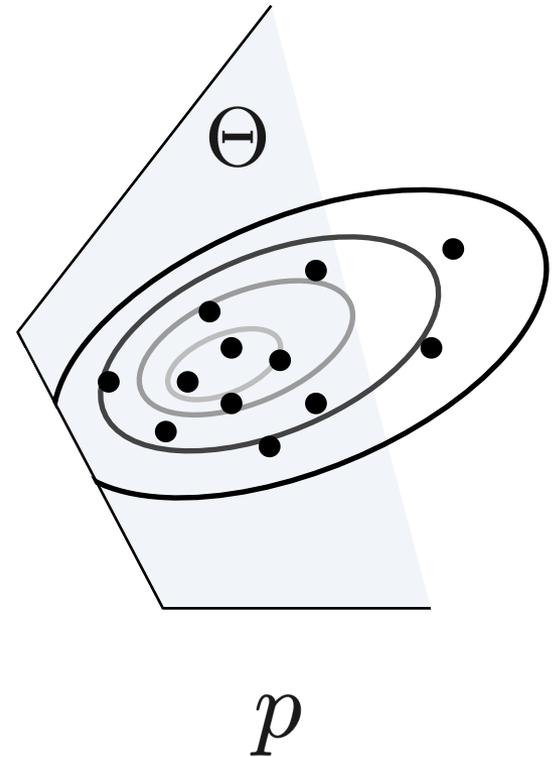
$$d\eta_t = -\nabla f(\theta_t)dt, \quad \theta_t = \nabla\psi^*(\eta_t)$$

Equivalent Riemannian gradient flow: $d\theta_t = -\nabla^2\psi(\theta_t)^{-1}\nabla f(\theta_t)dt$

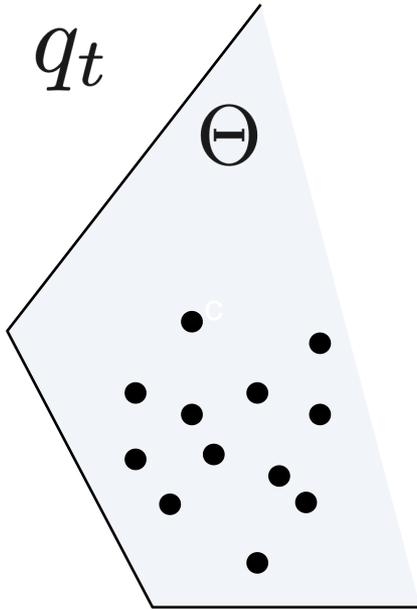
Mirrored Dynamics



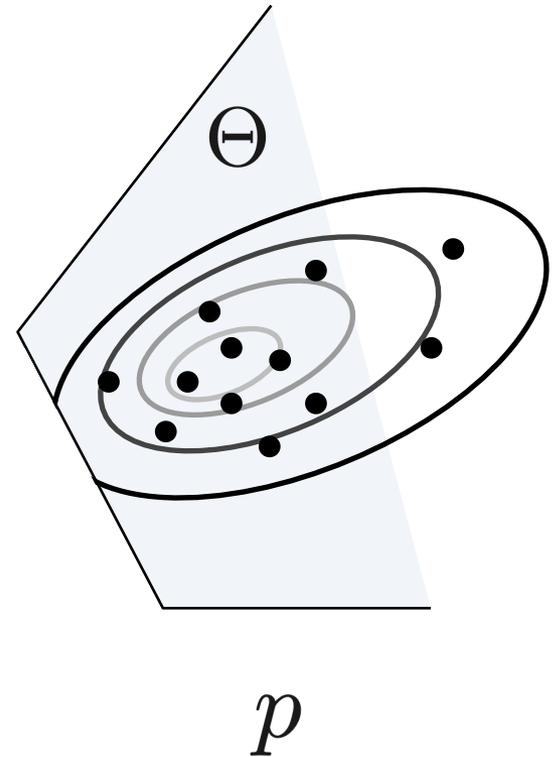
$$d\eta_t = g_t(\theta_t)dt,$$
$$\theta_t = \nabla\psi^*(\eta_t)$$



Mirrored Dynamics

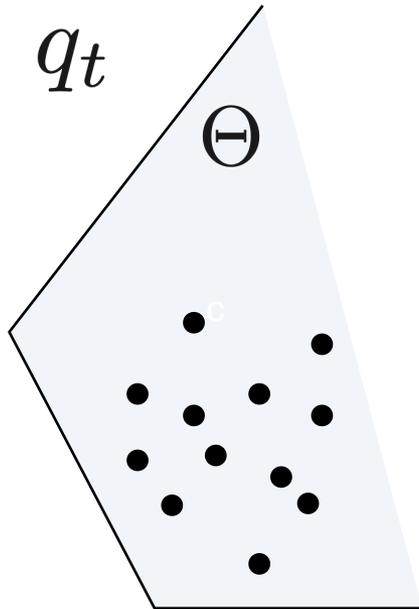


$$d\theta_t = \nabla^2 \psi(\theta_t)^{-1} g(\theta_t) dt$$

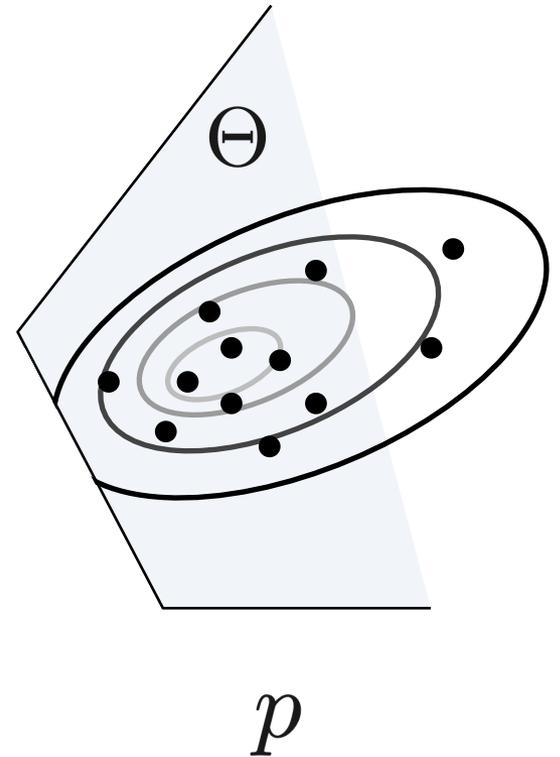


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Mirrored Stein Operator

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$$(\mathcal{M}_{p,\psi} g)(\theta) = g(\theta)^\top \nabla^2 \psi(\theta)^{-1} \nabla \log p(\theta) + \nabla \cdot (\nabla^2 \psi(\theta)^{-1} g(\theta))$$

A Stein Operator for Constrained Targets

Mirrored Stein Operator*

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Proposition 1 (informal) $\mathcal{M}_{p,\psi}$ generates mean-zero functions

under p if

$$\int_{\partial\Theta} p(\theta) \|\nabla^2 \psi(\theta)^{-1} n(\theta)\|_2 d\theta = 0$$

and $g \in C^1$ is bounded Lipschitz.

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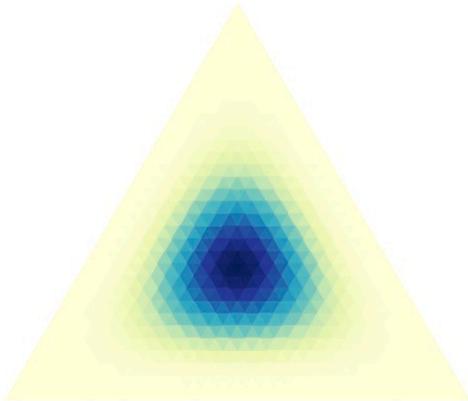
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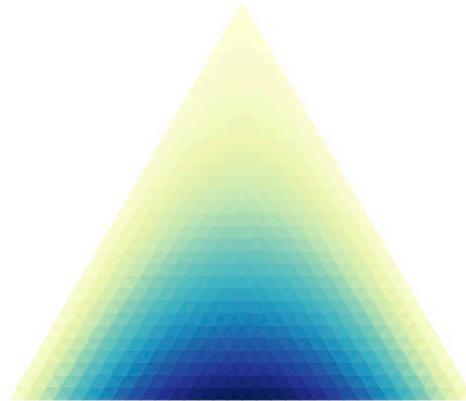
Intuitively, we expect $\nabla^2 \psi(\theta)^{-1}$ to **cancel the growth** of p at the boundary.

Example: The Dirichlet Distribution

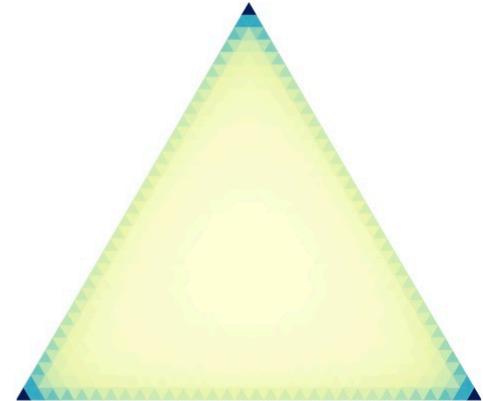
5, 5, 5



2, 2, 1



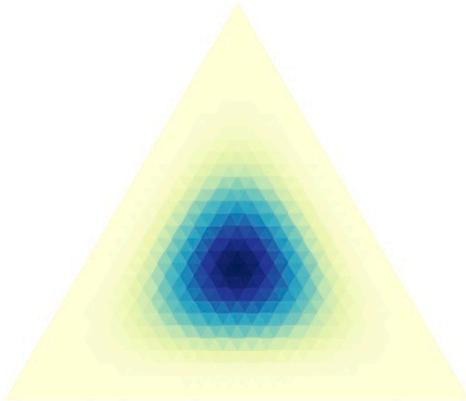
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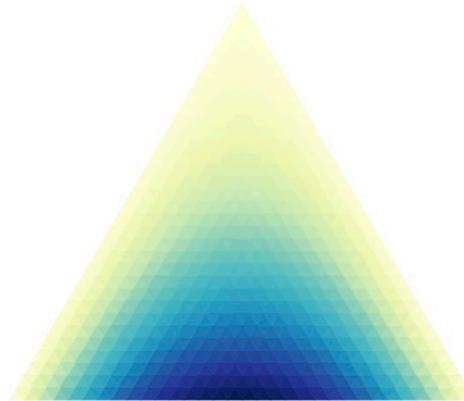
$$p(\theta) \propto \prod_{j=1}^{d+1} \theta_j^{\alpha_j - 1} \begin{cases} \alpha_j < 1 : \theta_j \rightarrow 0, \theta_{-j} = \frac{1 - \theta_j}{d} \Rightarrow p(\theta) \rightarrow \infty, \\ \alpha_j = 1 : \theta_j \rightarrow 0, \theta_{-j} = \frac{1 - \theta_j}{d} \Rightarrow p(\theta) > 0. \end{cases}$$

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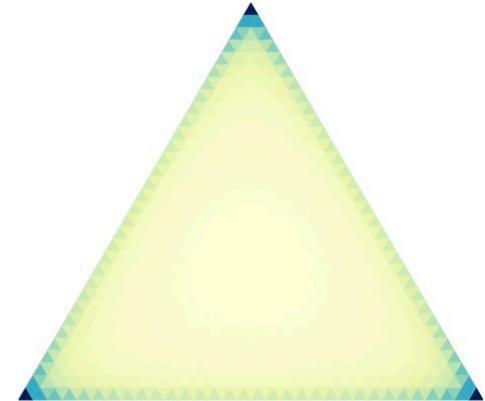
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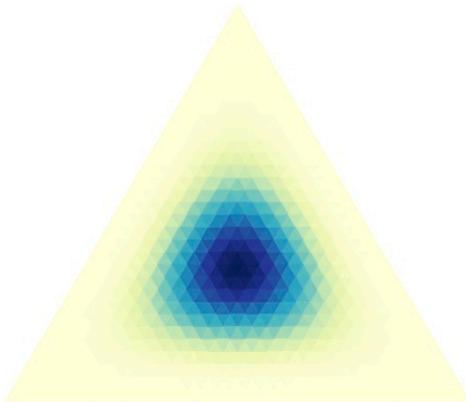
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Negative entropy $\psi(\theta) = \sum_{j=1}^{d+1} \theta_j \log \theta_j$ satisfies the boundary condition

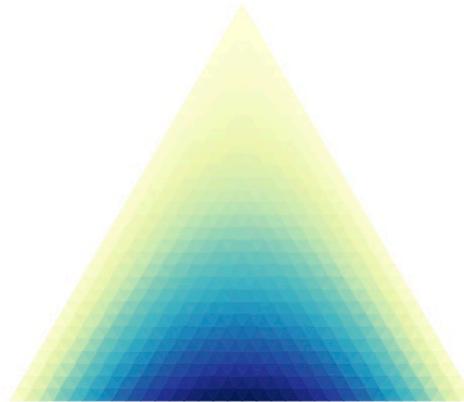
$$\int_{\partial\Theta} p(\theta) \|\nabla^2 \psi(\theta)^{-1} n(\theta)\|_2 d\theta = 0.$$

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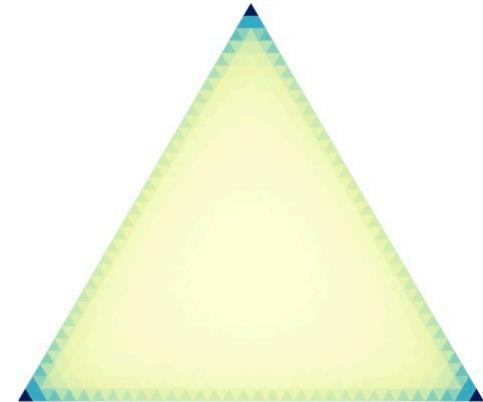
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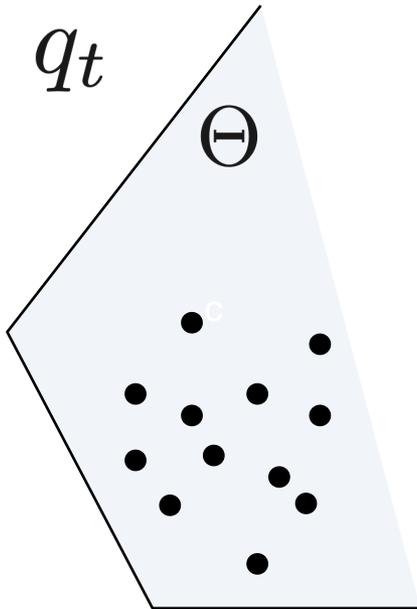


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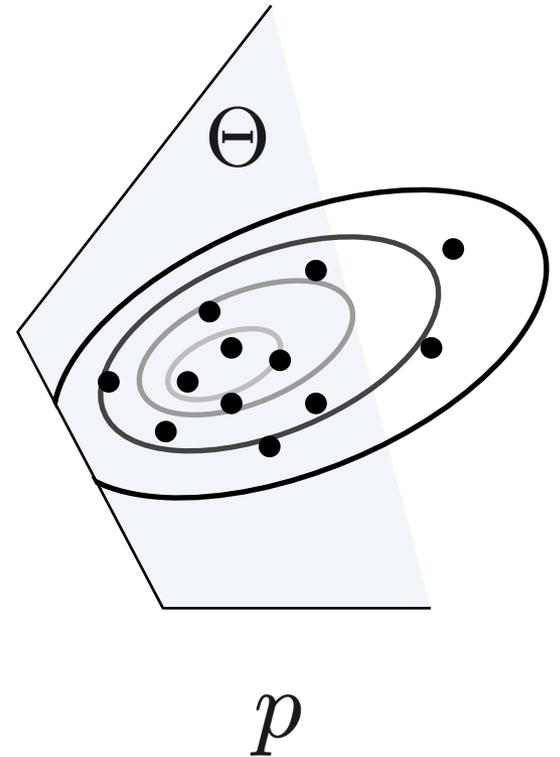
Negative entropy $\psi(\theta) = \sum_{j=1}^{d+1} \theta_j \log \theta_j$ satisfies the boundary condition $\nabla^2 \psi(\theta)^{-1} = \text{diag}(\theta) - \theta\theta^\top$

$$\int_{\partial\Theta} p(\theta) \|\nabla^2 \psi(\theta)^{-1} n(\theta)\|_2 d\theta = 0.$$

Mirrored Dynamics



$$d\theta_t = \nabla^2 \psi(\theta_t)^{-1} g(\theta_t) dt$$



Mirrored Stein Operator

$$\frac{d}{dt} \text{KL}(q_t \| p) = -\mathbb{E}_{q_t} [(\mathcal{M}_{p,\psi} g_t)(\theta)]$$

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Two Algorithms

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Choosing optimal g_t in

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- the RKHS of a fixed kernel
 - **Mirrored SVGD**: SVGD in the η space.
 - $n = 1$: **GD** on $-\log p_H(\eta)$.

Two Algorithms

Choosing optimal g_t in

- the RKHS of a fixed kernel
 - **Mirrored SVGD**: SVGD in the η space.
 - $n = 1$: **GD** on $-\log p_H(\eta)$.
- the RKHS of an adaptive kernel that incorporates the geometry
 - **Stein Variational Mirror Descent** (SVMD)
 - $n = 1$: **Mirror Descent** on $-\log p(\theta)$.

Mirrored SVGD (MSVGD)

Theorem 4 If $K(\theta, \theta') = k(\theta, \theta')I$, then the optimal mirrored updates can alternatively be expressed as

$$g_{q_t, kI}^*(\theta_t) = \mathbb{E}_{q_t, H} [k_\psi(\eta, \eta_t) \nabla \log p_H(\eta) + \nabla_\eta k_\psi(\eta, \eta_t)].$$

where $k_\psi(\eta, \eta') = k(\nabla\psi^*(\eta), \nabla\psi^*(\eta'))$

 transformed density of p
in dual space

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 transformed density of p
in dual space

- MSVGD is SVGD in η space with the **transformed kernel** k_ψ .
- When only a single particle is used ($n = 1$), Mirrored SVGD reduces to gradient ascent on the log transformed density $\log p_H(\eta)$.

Single Particle MSVGD is Not Mirror Descent

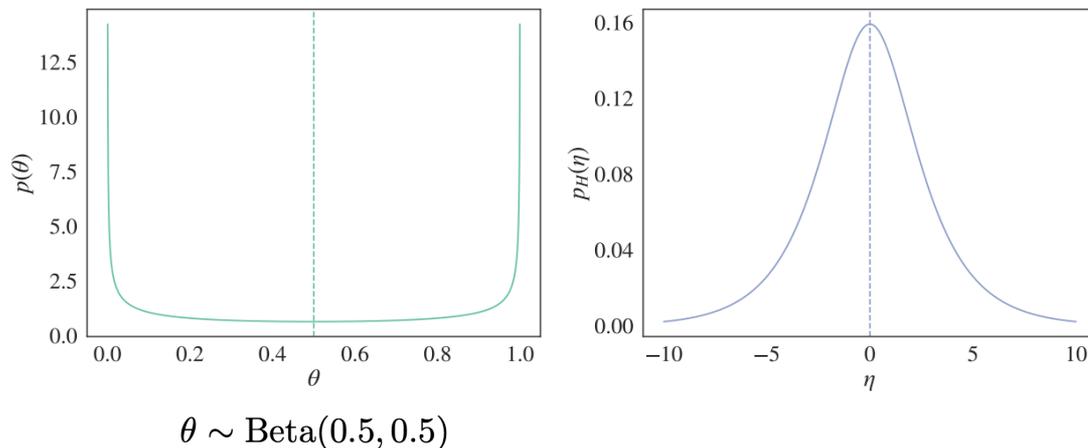
Still want an algorithm that reduces to mirror descent when $n = 1$?

- θ space is the space we are primarily interested in.
- Mode in θ space need not match mode in η space
- Using $\log p(\theta)$ to guide the evolution could work better if $p(\theta)$ is better behaved than $p_H(\eta)$.

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Stein Variational Mirror Descent (SVMD)

Key idea: Construct an **adaptive kernel** that

- ① incorporates the metric induced by ψ
- ② evolves with q_t

Definition (Kernels for SVMD)

Given a reference kernel k , we write it in Mercer's representation:

$$k(\theta, \theta') = \sum_{i \geq 1} \lambda_i u_i(\theta) u_i(\theta'),$$

where u_i is an eigenfunction satisfying:

$$\mathbb{E}_{q_t(\theta')} [k(\theta, \theta') u_i(\theta')] = \lambda_i u_i(\theta).$$

Kernels for SVMD:

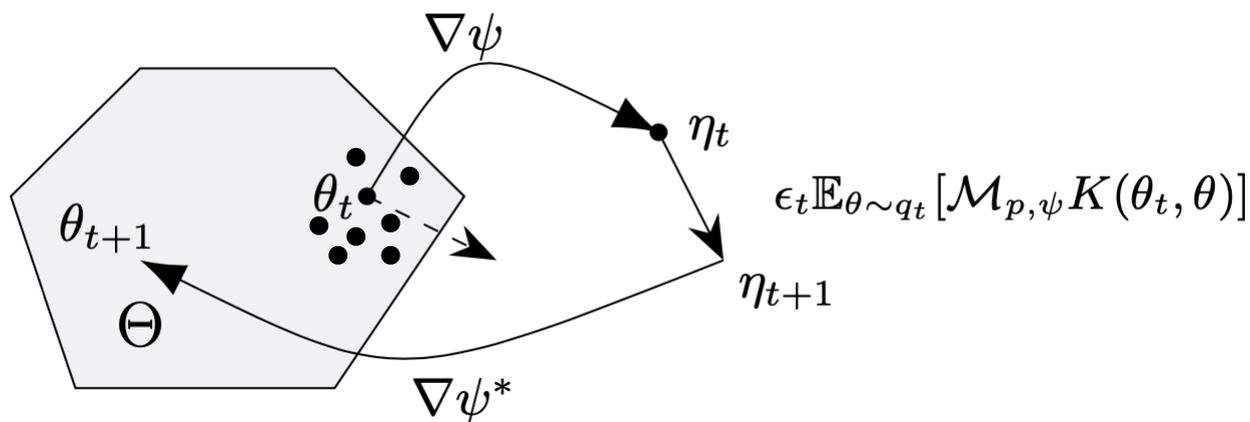
$$k^{1/2}(\theta, \theta') \triangleq \sum_{i \geq 1} \lambda_i^{1/2} u_i(\theta) u_i(\theta')$$

$$K_{\psi, t}(\theta, \theta') \triangleq \mathbb{E}_{\theta_t \sim q_t} [k^{1/2}(\theta, \theta_t) \nabla^2 \psi(\theta_t) k^{1/2}(\theta_t, \theta')]$$

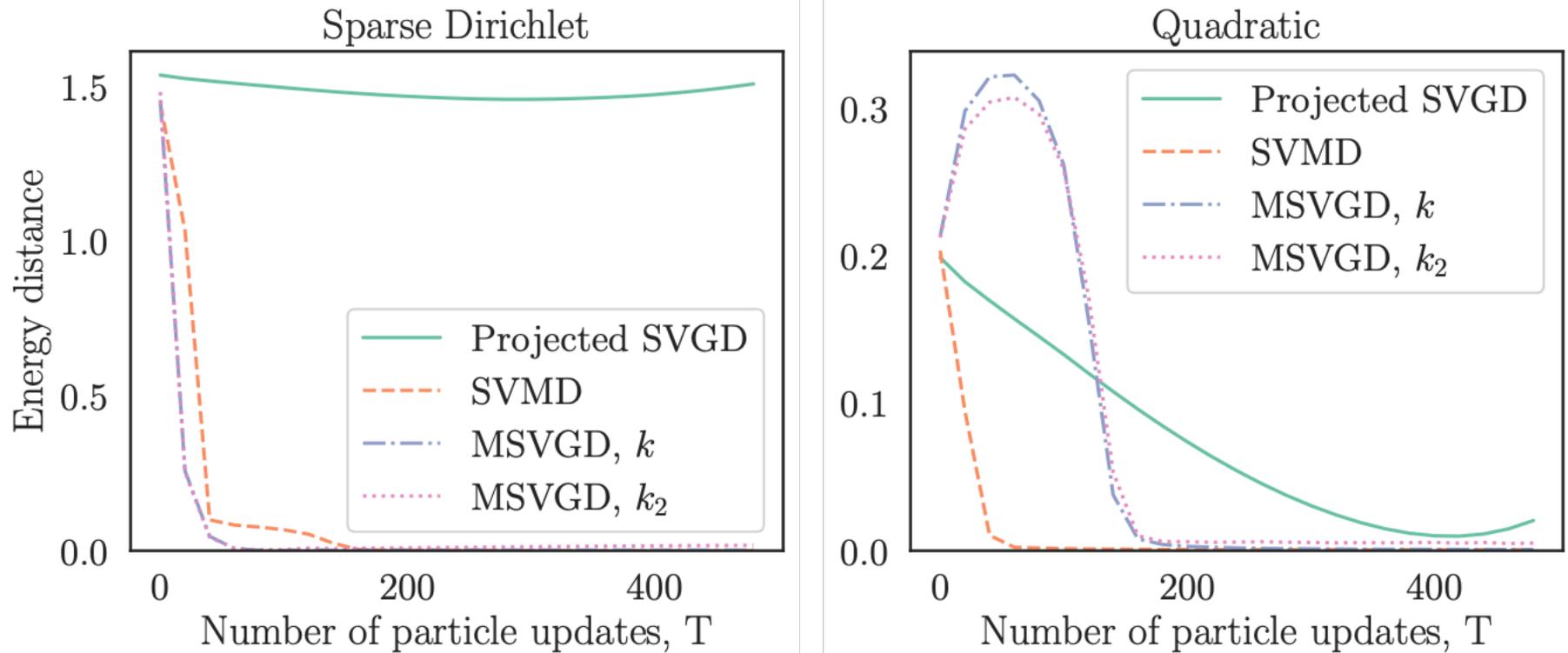
A Multi-Particle Generalization of Mirror Descent

If $n = 1$, then one-step of SVMD becomes

$$\eta_{t+1} = \eta_t + \epsilon_t (k(\theta_t, \theta_t) \nabla \log p(\theta_t) + \nabla k(\theta_t, \theta_t)),$$
$$\theta_{t+1} = \nabla \psi^*(\eta_{t+1}).$$



Approximation Quality on the Simplex



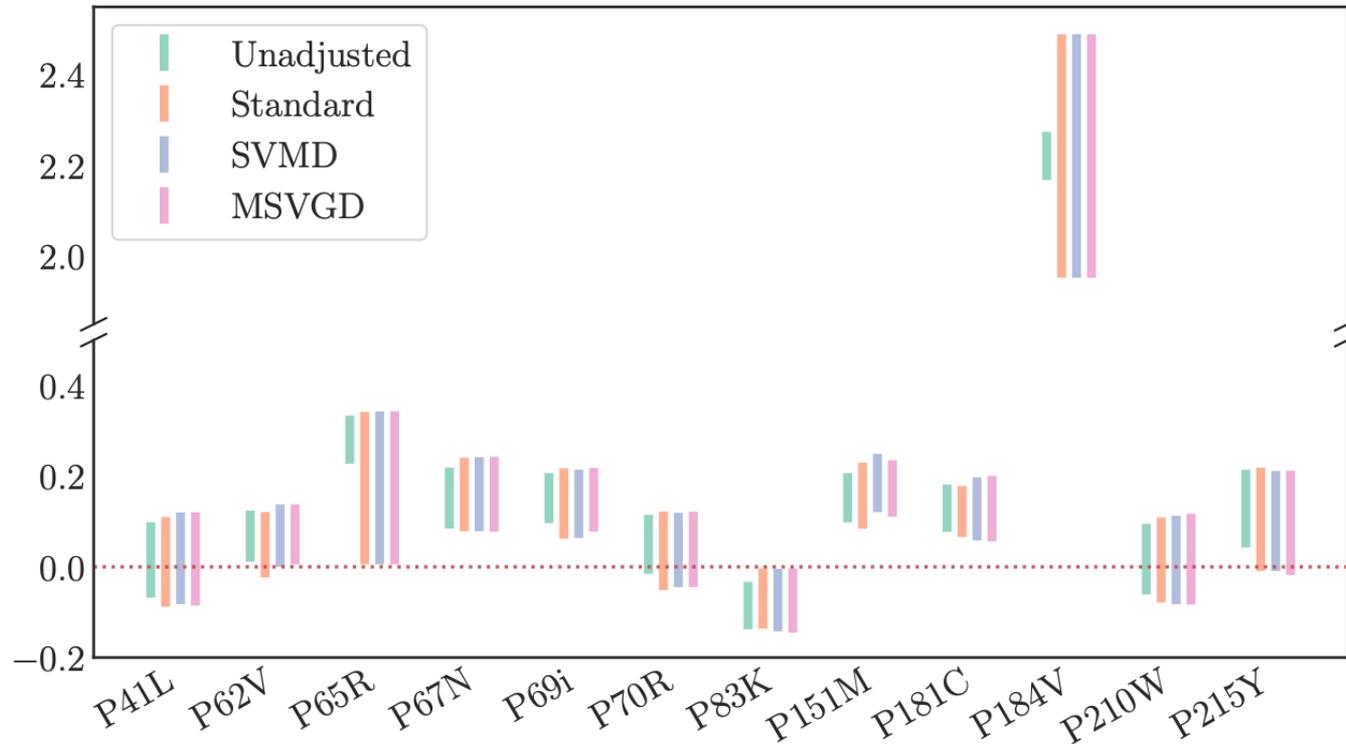
Quality of 50-particle approximations to 20-dimensional distributions on the simplex.

Application: Post-Selection Inference

Task: Generate valid confidence intervals (CIs) for parameters after data-driven model (feature) selection.

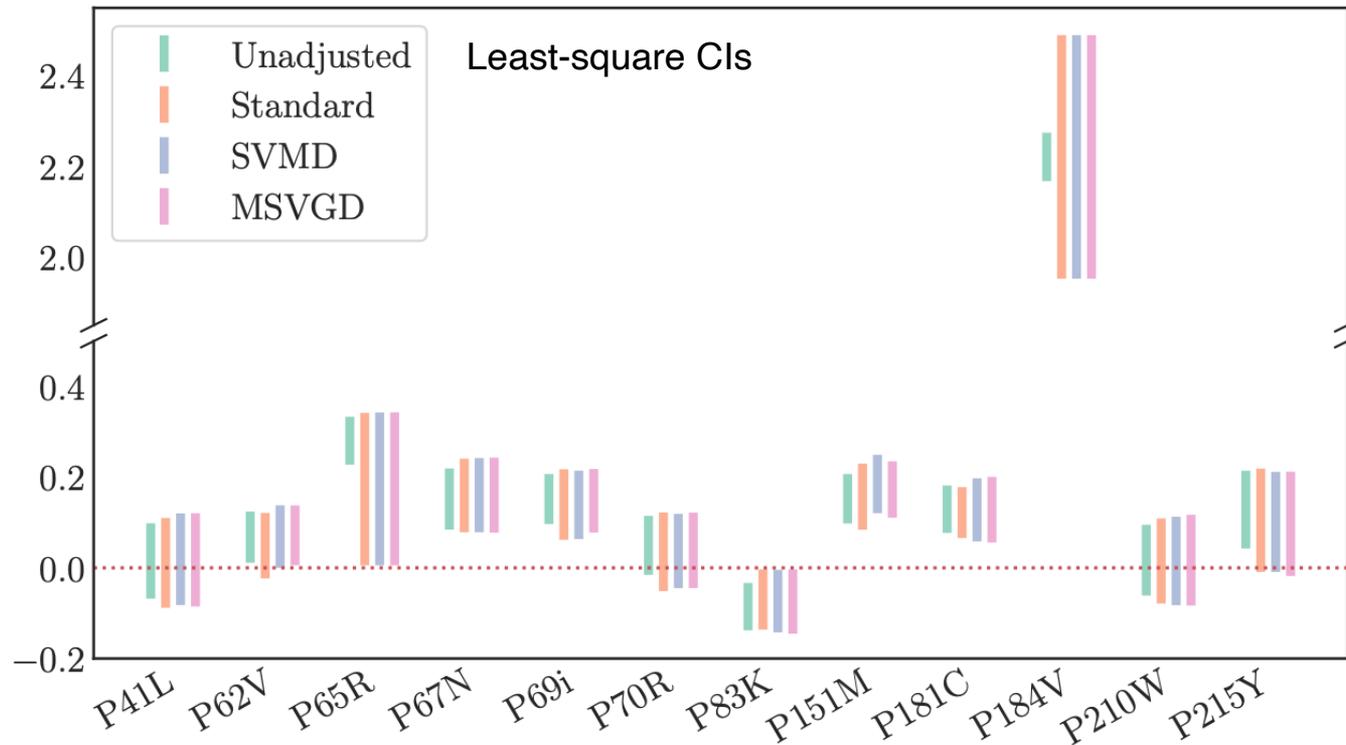
- Need to condition on the selection event.
- Target distributions are log-concave and have constrained support.

Application: Post-Selection Inference



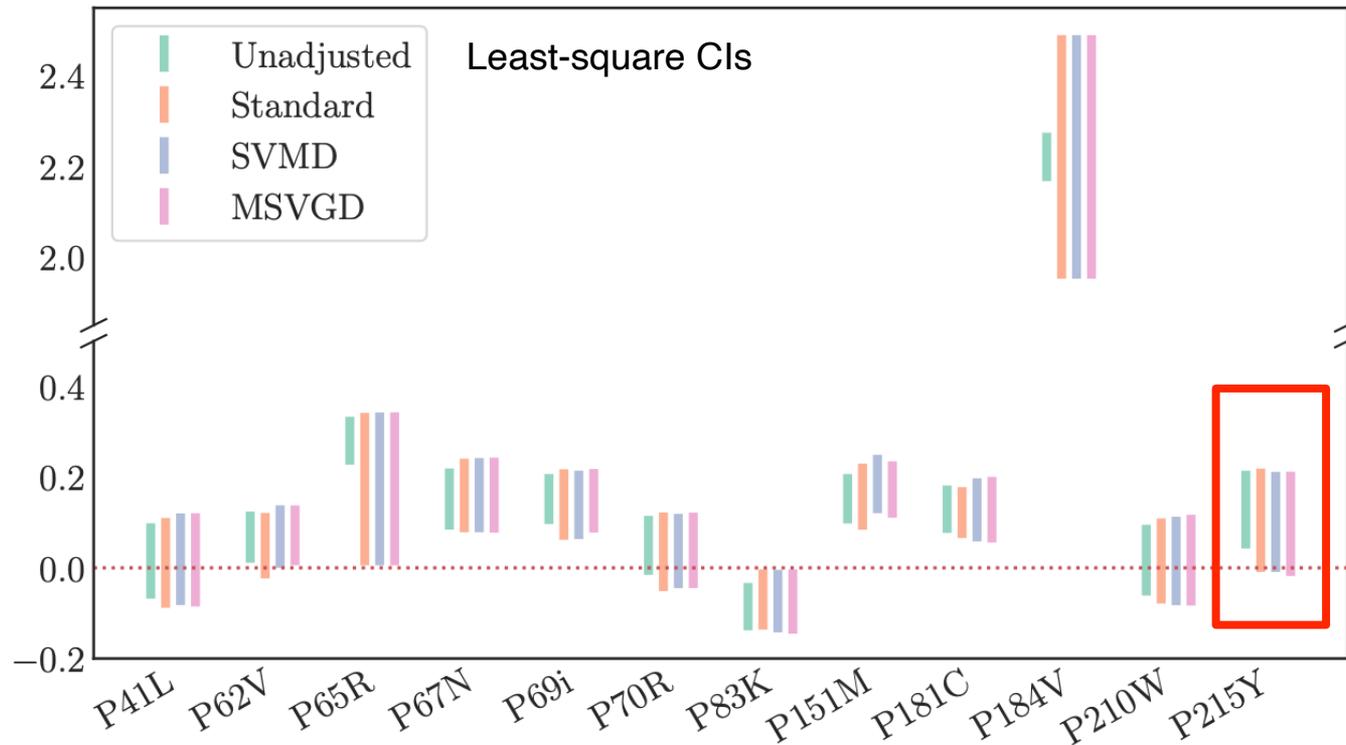
Unadjusted and post-selection CIs for the mutations selected by the randomized Lasso as candidates for HIV-1 drug resistance.

Application: Post-Selection Inference



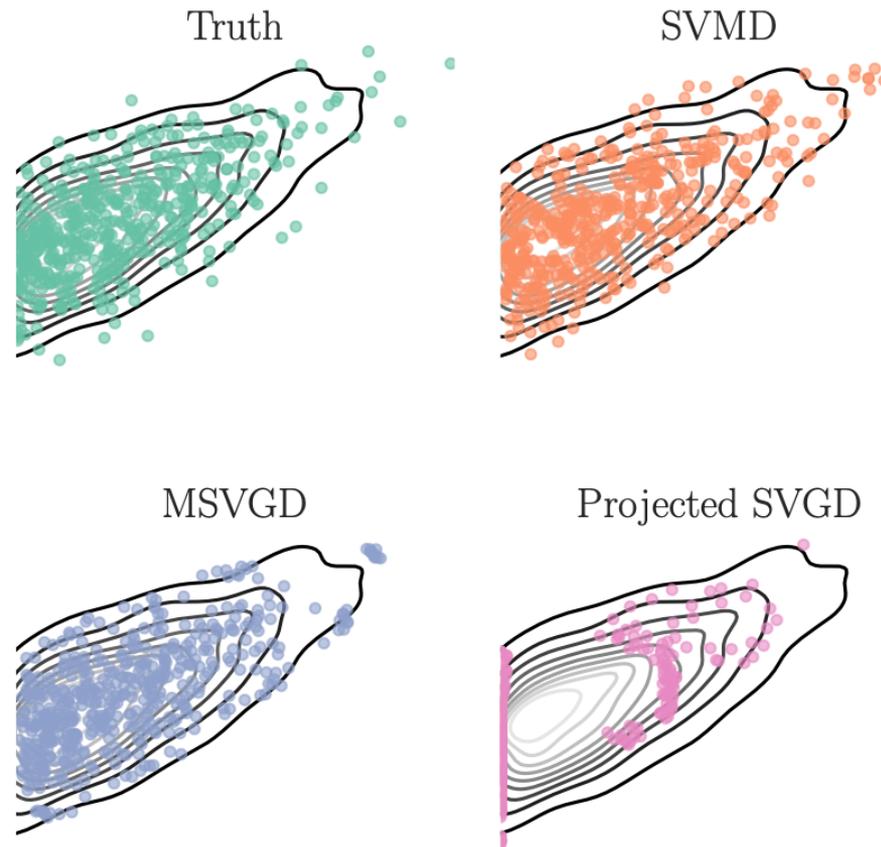
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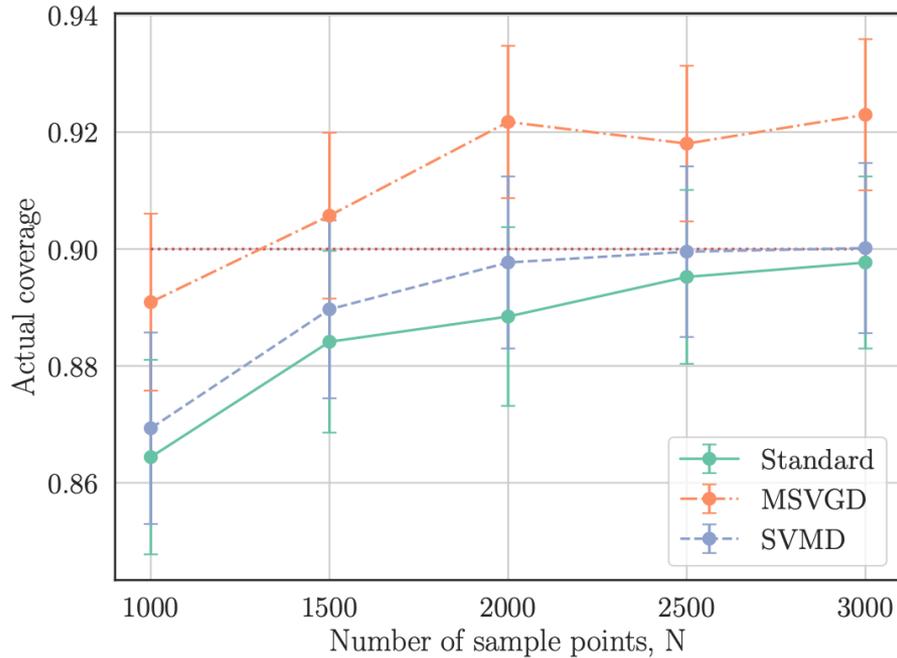
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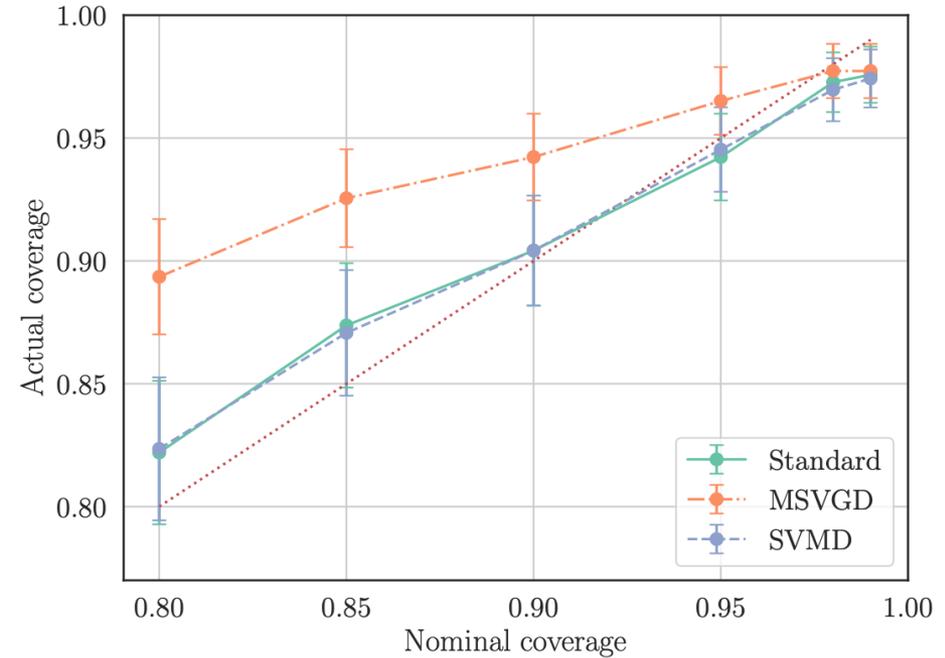


A 2D selective density example.

Application: Post-Selection Inference



Nominal Coverage: 0.9



5000 sample points

Coverage of post-selection CIs.

Convergence Results

- ① Convergence of mirrored updates as $n \rightarrow \infty$.
- ② Infinite-particle mirrored Stein updates decrease KL with sufficiently small step size and drive Mirrored Kernel Stein Discrepancy (MKSD) to 0.
- ③ MKSD determines weak convergence under suitable conditions.

Convergence Results

① Convergence of mirrored updates as $n \rightarrow \infty$.

Theorem Suppose $q_{0,H}^n = \frac{1}{n} \sum_{i=1}^n \delta_{\eta_0^i}$ satisfying $W_1(q_{0,H}^n, q_{0,H}^\infty) \rightarrow 0$. Define the η -induced kernel $K_{\nabla\psi^*,t}(\eta, \eta') := K_t(\nabla\psi^*(\eta), \nabla\psi^*(\eta'))$. If, for some $c_1, c_2 > 0$:

$$\|\nabla(K_{\eta,t}(\cdot, \eta) \nabla \log p_H(\eta) + \nabla \cdot K_{\eta,t}(\cdot, \eta))\|_{\text{op}} \leq c_1(1 + \|\eta\|_2),$$

$$\|\nabla(K_{\eta,t}(\eta', \cdot) \nabla \log p_H(\cdot) + \nabla \cdot K_{\eta,t}(\eta', \cdot))\|_{\text{op}} \leq c_2(1 + \|\eta'\|_2),$$

Then $W_1(q_{t,H}^n, q_{t,H}^\infty) \rightarrow 0$ for each round of t

Convergence Results

② Infinite-particle mirrored Stein updates decrease KL with sufficiently small step size and drive Mirrored Kernel Stein Discrepancy (MKSD) to 0.

Theorem Assume $\kappa_1 := \sup_{\theta} \|K_t(\theta, \theta)\|_{\text{op}} < \infty$, and $\kappa_2 := \sum_{i=1}^d \sup_{\theta} \|\nabla_{i,d+i}^2 K_t(\theta, \theta)\|_{\text{op}} < \infty$, $\nabla \log p_H$ is L -Lipschitz, and ψ is α -strongly convex. If ϵ_t is sufficiently small, then

$$\text{KL}(q_{t+1}^{\infty} \| p) - \text{KL}(q_t^{\infty} \| p) \leq - \left(\epsilon_t - \left(\frac{L\kappa_1}{2} + \frac{2\kappa_2}{\alpha^2} \right) \epsilon_t^2 \right) \text{MKSD}_{K_t}(q_t^{\infty} \| p)^2.$$

$$\text{MSD}(q, p, \mathcal{G}) \triangleq \sup_{g \in \mathcal{G}} \mathbb{E}_q[(\mathcal{M}_{p, \psi} g)(\theta)] \quad \text{and} \quad \text{MKSD}_K(q, p) \triangleq \text{MSD}(q, p, \mathcal{B}_{\mathcal{H}_K}).$$

From Constrained to Unconstrained Targets

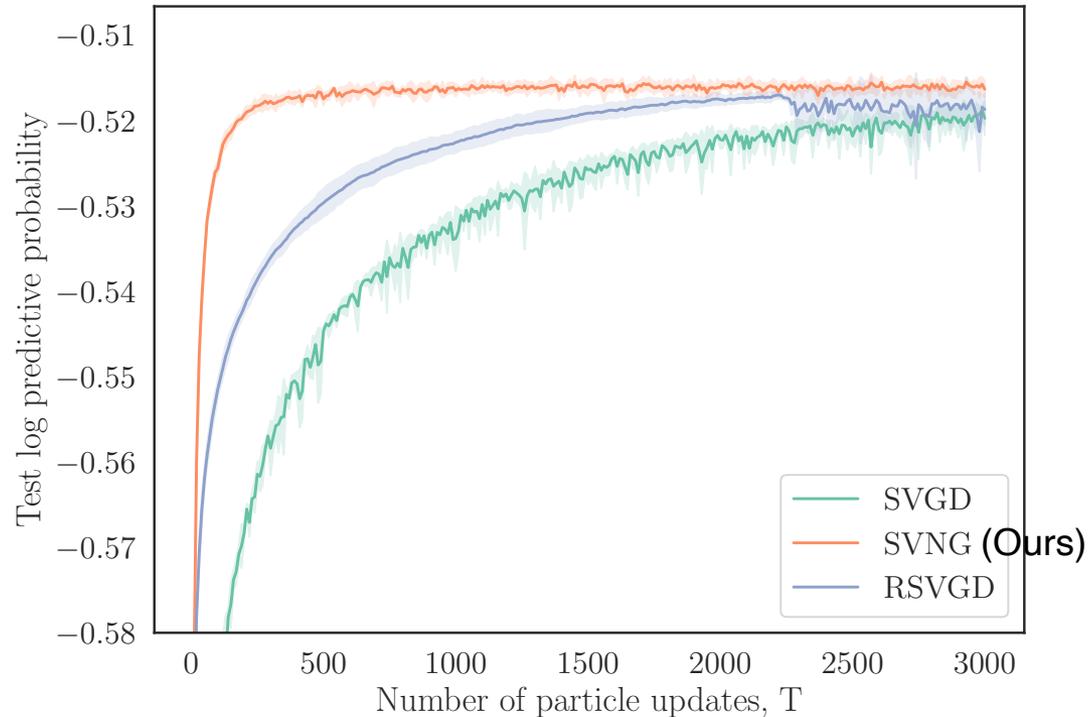
Continuous Time	Discretization
<p>Mirror flow:</p> $d\eta_t = -\nabla f(\theta_t)dt,$ $\theta_t = \nabla\psi^*(\eta_t)$	<p>Mirror descent</p>
<p>Riemannian gradient flow with metric tensor $\nabla^2\psi$:</p> $d\theta_t = -\nabla^2\psi(\theta_t)^{-1}\nabla f(\theta_t)dt$	<p>Natural gradient descent with metric tensor $\nabla^2\psi$</p>

Stein Variational Natural Gradient (SVNG)

- Replacing $\nabla^2 \psi(\cdot)$ in SVMMD with a general metric tensor
- In Bayesian inference $p(\theta) \propto \pi(\theta)\pi(y|\theta)$, it is common to choose

$$\text{FIM: } G(\theta) = \mathbb{E}_{\pi(y|\theta)}[\nabla \log \pi(y|\theta) \nabla \log \pi(y|\theta)^\top]$$

Exploiting Geometry in Bayesian Inference



Posterior inference for large-scale Bayesian Logistic Regression
581,012 datapoints, $d = 54$

Takeaways

- A new family of particle evolution samplers suitable for **constrained domains** and **non-Euclidean geometries**.
- SVMD is a multi-particle generalization of **mirror descent** for constrained sampling problems
- SVNG can exploit the geometry of unconstrained sampling problems with user-specified metric tensors.

Future Work

- Complexity can be cubic w.r.t. the number of particles.
- Where you need mirror descent before, would it benefit from using a variant that is aware of uncertainty?

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